HOSSAM GHANEM

(25) 8.3 Quadratic Expressions

	كيف تحول المقدار الثلاثي إلي مربع كامل	
	الحالة الأولى : x^2 موجبة	x^2-6x-7
1	ضع $x^2, -6x$ في قوس مع ترك فراغ في القوس كما هو مبين	$(x^2-6x \quad)-7$
2	اقسم معامل x على 2 ثم ربع الناتج وضعه في الفراغ	$(x^2-6x+9)-7$
3	ضع نفس العدد بإشارة سالب خارج القوس	$(x^2-6x+9)-7-9$
4	حلل القوس مربع كامل " جذر الأول ثم إشارة الأوسط ثم جذر الأخير الكل تربيع" ثم اجمع العددين خارج القوس	$(x-3)^2-16$

	كيف تحول المقدار الثلاثي إلي مربع كامل	
	الحالة الثانية : x^2 سالبة	$9-8x-x^2$
1	ضع العدد أولا ثم أترك مسافة وخذ إشارة سالب ثم افتح قوس وضع $x^2, -8x$ في قوس بإشارة مخالفة مع ترك فراغ في القوس كما هو مبين	$9 \quad -(x^2-8x \quad)$
2	اقسم معامل x على 2 ثم ربع الناتج وضعه في الفراغ	$9 \quad -(x^2-8x+16)$
3	ضع نفس العدد بإشارة موجب خارج القوس	$9+16-(x^2-8x+16)$
4	اجمع العددين خارج القوس ثم حلل القوس مربع كامل " جذر الأول ثم إشارة الأوسط ثم جذر الأخير الكل تربيع"	$25-(x-4)^2$

	كيف تحل مسألة Quadratic Expressions	
1	نلاحظ في هذه المسألة وجود x^2, x تحت الجذر ولذلك نكمل المربع	$I=\int \frac{1}{(x-2)\sqrt{x^2-4x+3}}dx$
2	كمل المربع كما سبق في الجدول الأول	$x^2-4x+3=$ $(x^2-4x+4)+3-4=$ $(x-2)^2-1$
3	فيصبح التكامل	$I=\int \frac{1}{(x-2)\sqrt{(x-2)^2-1}}dx$
4	استخدم التعويض	$x-2=t$
5	اشتق لتحصل على dt	$dx=dt$
6	يصبح التكامل	$I=\int \frac{1}{t\sqrt{t^2-1}}dx=\sec^{-1}t+c$
7	عوض عن t مرة أخرى	$I=\sec^{-1}(x-2)+c$

Example 1 Evaluate the following ($2\,{}^{1}/_{2}$ points) $\int \frac{1}{x^2+2x+2}\,dx$ 56 11 December 2011

Solution

$$x^2+2x+2=(x^2+2x+1)-1+2=(x+1)^2+1$$

$$I=\int \frac{1}{x^2+2x+2}\,dx=\int \frac{1}{(x+1)^2+1}\,dx$$

Let $t=x+1$ $\quad dt=dx$

$$I=\int \frac{1}{t^2+1}\,dx=\tan^{-1}t+c=\tan^{-1}(x+1)+c$$

Example 2 Evaluate $\int \frac{x}{(6x-8-x^2)^{\frac{3}{2}}}\,dx$ 35 December 2004

Solution

$$6x-8-x^2=-8-(x^2-6x)=-8+9-(x^2-6x+9)=1-(x-3)^2$$

$$I=\int \frac{x}{(6x-8-x^2)^{\frac{3}{2}}}\,dx=\int \frac{x}{[1-(x-3)^2]^{\frac{3}{2}}}\,dx$$

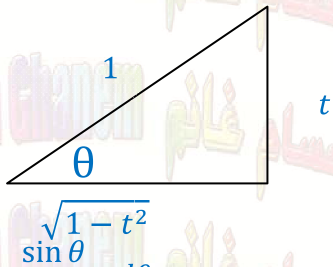
Let $t=x-3 \Rightarrow x=t+3 \Rightarrow dt=dx$

$$I=\int \frac{t+3}{(1-t^2)^{\frac{3}{2}}}\,dt$$

$t=\sin\theta \qquad dt=\cos\theta\ d\theta$

$$\sin\theta=\frac{t}{1}$$

$$\theta=\sin^{-1}t$$



$$I=\int \frac{\sin\theta+3}{(1-\sin^2\theta)^{\frac{3}{2}}}\,dt=\int \frac{3+\sin\theta}{\cos^3\theta}\cos\theta\ d\theta=\int \frac{3+\sin\theta}{\cos^2\theta}\ d\theta=\int \frac{3}{\cos^2\theta}+\frac{\sin\theta}{\cos^2\theta}\ d\theta$$

$$=\int (3\sec^2\theta+\tan\theta\sec\theta)\ d\theta=3\tan\theta+\sec\theta+c$$

$$=3\,\frac{t}{\sqrt{1-t^2}}+\frac{1}{\sqrt{1-t^2}}+c=\frac{3(x-3)}{\sqrt{1-(x-3)^2}}+\frac{1}{\sqrt{1-(x-3)^2}}+c$$

Example 3 Evaluate $\int x\sqrt{8-2x-x^2}\,dx$ 36 June 2005

Solution

$8-2x-x^2 = 8-(x^2+2x) = 8+1-(x^2+2x+1) = 9-(x+1)^2$

$I = \int x\sqrt{8-2x-x^2}\,dx = \int x\sqrt{9-(x+1)^2}\,dx$

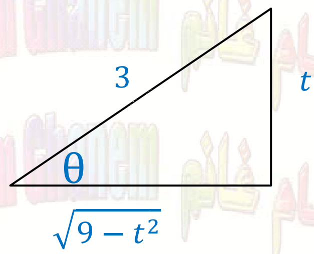
Let $t = x+1 \quad \Rightarrow dt = dx \quad \Rightarrow x = t-1$

$I = \int (t-1)\sqrt{9-t^2}\,dt$

$t = 3\sin\theta \qquad dt = 3\cos\theta\;d\theta$

$\sin\theta = \frac{t}{3}$

$\theta = \sin^{-1} t$



$= \int (3\sin\theta-1)\sqrt{9-9\sin^2\theta}\;3\cos\theta\;d\theta = \int (3\sin\theta-1)3\cos\theta\cdot 3\cos\theta\;d\theta$

$= \int (3\sin\theta-1)\cdot 9\cos^2\theta\;d\theta = \int 27\cos^2\theta\sin\theta - 9\cos^2\theta\;d\theta$

$= \int 27\cos^2\theta\sin\theta - \frac{9}{2}(1+\cos 2\theta)\,d\theta = 9\cos^3\theta - \frac{9}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)+c$

$= 9\cos^3\theta - \frac{9}{2}(\theta+\sin\theta\cos\theta)+c = 9\cos^3\theta - \frac{9}{2}\theta - \frac{9}{2}\sin\theta\cos\theta + c$

$= 9\left(\frac{\sqrt{9-t^2}}{3}\right)^3 - \frac{9}{2}\sin^{-1}\left(\frac{t}{3}\right) - \frac{9}{2}\cdot\frac{t}{3}\cdot\frac{\sqrt{9-t^2}}{3}+c$

$= 9\left(\frac{\sqrt{9-(x+1)^2}}{3}\right)^3 - \frac{9}{2}\sin^{-1}\left(\frac{x+1}{3}\right) - \frac{9}{2}\cdot\frac{x+1}{3}\cdot\frac{\sqrt{9-(x+1)^2}}{3}+c$

$= \frac{1}{3}\left(\sqrt{8-2x-x^2}\right)^3 - \sin^{-1}\left(\frac{x+1}{3}\right) - \frac{1}{2}(x+1)\sqrt{8-2x-x^2}+c$

Example 4 Evaluate $\int \frac{(x^2+2x-3)^{\frac{3}{2}}}{x+1}dx$ 40 May 2006

Solution

$x^2+2x-3=(x^2+2x+1)-1-3=(x+1)^2-4$

$$I=\int \frac{(x^2+2x-3)^{\frac{3}{2}}}{x+1}dx=\int \frac{[(x^2+1)^2-4]^{\frac{3}{2}}}{x+1}dx$$

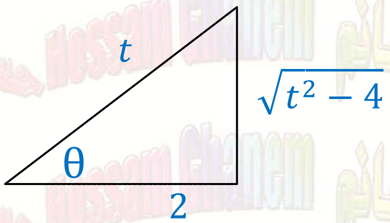
Let $t=x+1$ $dt=dx$

$$I=\int \frac{(t^2-4)^{\frac{3}{2}}}{t}dx$$

$t=2\sec\theta$ $dt=2\sec\theta\tan\theta\ \ d\theta$

$\sec\theta=\frac{t}{2}$

$\theta=\sec^{-1}\left(\frac{t}{2}\right)$



$$I=\int \frac{(4\sec^2\theta-4)^{\frac{3}{2}}}{2\sec\theta}\ 2\sec\theta\tan\theta\ \ d\theta$$

$$=\int \frac{2^3\tan^3\theta}{2\sec\theta}\ 2\sec\theta\tan\theta\ \ d\theta=2^3\int\tan^4\theta\ \ d\theta=8\int(\tan^2\theta)^2\,d\theta=8\int(\sec^2\theta-1)^2\ \ d\theta$$

$$I=8\int(\sec^4\theta-2\sec^2\theta+1)\ \ d\theta$$

$$I_1=\int\sec^4\theta\ \ d\theta=\int\sec^2\theta\ \cdot\sec^2\theta\ \ d\theta=\int(1+\tan^2\theta)\ \sec^2\theta\ \ d\theta$$

Let $u=\tan\theta$ ⇨ $du=\sec^2\theta\ \ d\theta$

$$I_1=\int(1+u^2)\,du=u+\frac{1}{3}\ u^3+c_1=\tan\theta+\frac{1}{3}\ \tan^3\theta+c_1$$

$$I=8\tan\theta+\frac{8}{3}\ \tan^3\theta-16\tan\theta+8\theta+c=\frac{8}{3}\ \tan^3\theta-8\tan\theta+8\theta+c$$

$$=\frac{8}{3}\left(\frac{\sqrt{t^2-4}}{2}\right)^3-8\left(\frac{\sqrt{t^2-4}}{2}\right)+8\sec^{-1}\left(\frac{t}{2}\right)$$

$$=\frac{8}{3}\left(\frac{\sqrt{x^2+2x-3}}{2}\right)^3-8\left(\frac{\sqrt{x^2+2x-3}}{2}\right)+8\sec^{-1}\left(\frac{x+1}{2}\right)+c$$

Example 5 Evaluate $\displaystyle\int \frac{\sqrt{e^{2x}-2e^{x}-8}}{1-e^{-x}}\,dx$

Solution

$$I=\int \frac{\sqrt{e^{2x}-2e^{x}-8}}{1-e^{-x}}\,dx=\int \frac{\sqrt{e^{2x}-2e^{x}-8}}{e^{x}-1}\cdot e^{x}dx$$

بالضرب في e^x بسط ومقام

Let $t=e^x \quad \Rightarrow dt=e^x\,dt$

$$I=\int \frac{\sqrt{t^2-2t-8}}{t-1}dt$$

$$t^2-2t-8=(t^2-2t+1)-1-8=(t-1)^2-9$$

$$I=\int \frac{\sqrt{(t-1)^2-9}}{t-1}dx$$

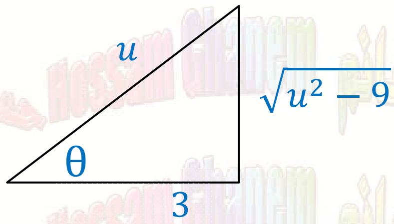
Let $u=t-1 \qquad du=dt$

$$I=\int \frac{\sqrt{u^2-9}}{u}dx$$

$u=3\sec\theta \qquad\qquad du=3\sec\theta\tan\theta\;d\theta$

$$\sec\theta=\frac{u}{2}$$

$$\theta=\sec^{-1}\left(\frac{u}{3}\right)$$



$$I=\int \frac{\sqrt{9\sec^2\theta-9}}{3\sec\theta}\;3\sec\theta\tan\theta\;d\theta$$

$$=3\int \tan^2\theta\;d\theta=3\int(\sec^2\theta-1)\,d\theta=3\tan\theta-3\theta+c$$

$$=3\cdot\frac{\sqrt{u^2-9}}{3}-3\sec^{-1}\left(\frac{u}{3}\right)+c=3\cdot\frac{\sqrt{(t-1)^2-9}}{3}-3\sec^{-1}\left(\frac{t-1}{3}\right)+c$$

$$=\sqrt{(e^x-1)^2-9}-3\sec^{-1}\left(\frac{e^x-1}{3}\right)+c$$

Example 6 Evaluate $\int \frac{\sec x}{2\sec x + \tan x + 1}\,dx$ 46 July 2008

Solution

$$I = \int \frac{\sec x}{2\sec x + \tan x + 1}\,dx = \int \frac{1}{2 + \sin x + \cos x}\,dx$$ بالضرب في $\cos x$ بسط و مقام

$$\sin x = \frac{2u}{1+u^2} \qquad \cos x = \frac{1-u^2}{1+u^2} \qquad \tan x = \frac{2u}{1-u^2} \qquad dx = \frac{2}{1+u^2}\,du \qquad u = \tan\frac{x}{2}$$

$$I = \int \frac{1}{2 + \frac{2u}{1+u^2} + \frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}\,du = \int \frac{1}{2 + 2u^2 + 2u + 1 - u^2}\,du = 2\int \frac{1}{u^2 + 2u + 3}\,du$$

$$u^2 + 2u + 3 = (u^2 + 2u + 1) - 1 + 3 = (u+1)^2 + 2$$

$$I = 2\int \frac{1}{(u+1)^2 + 2}\,du$$

$$t = u + 1 \qquad dt = du$$

$$I = 2\int \frac{1}{t^2 + 2}\,du = 2 \cdot \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + c = \sqrt{2}\tan^{-1}\left(\frac{\tan\frac{x}{2} + 1}{\sqrt{2}}\right) + c$$

Homework

1	Evaluate the integral	$\int_2^4 \frac{2x+1}{\sqrt{6x-x^2-5}}dx$	37 June 2005
2	Evaluate the integral	$\int \frac{x^2+6x+9}{\sqrt{7-6x-x^2}}dx$	44 July 2007
3	Evaluate the integral	$\int \frac{x}{\sqrt{x^2+2x-3}}dx$	42 December 2006
4	Evaluate the integral	$\int \frac{\sqrt{x^2-2x-8}}{x-1}dx$	41 July 2006
5	(3 pts) Evaluate the following integrals	$\int \frac{\sqrt{4x-x^2-3}}{x-1}\,dx$	40 August 7 , 2011
6	Evaluate the integral	$\int \frac{x^2}{(9-x^2)^{\frac{3}{2}}}dx$	48 May 2009
7	Evaluate the integral	$\int \frac{2x-1}{\sqrt{x^2-6x+5}}dx$	49 July 2009
8	Evaluate the following. [3.5 pts.]	$\int x\sqrt{2x-x^2}\,dx$	51 May 13, 2010
9	Evaluate the following integral [3 marks e]	$\int \sqrt{6x-5-x^2}\,dx$	52 July 24, 2010
10	(3 pts.) Evaluate the following	$\int_0^{2-\sqrt{2}} \frac{x^2-4x+5}{(2-x)\sqrt{x^2-4x+3}}\,dx$	53 11 Dec. 2010
11	Evaluate the following integral	$\int \frac{x}{(x-2)\sqrt{x^2-4x+3}}\,dx$	54 12/05/2011
12	Evaluate the following integral	$\int \frac{x}{x^2-4x+8}\,dx$	35 January 24, 2010
13	Evaluate the following integrals. (3 pts)	$\int \frac{7x^2-4x-17}{x^3+2x^2+17x}\,dx$	55 July 23 , 2011
14	Evaluate the integral	$\int \frac{x}{(x^2-4x+8)^{\frac{3}{2}}}dx$	47 December 2008

Homework

1	Evaluate the integral	$\int \frac{(x-3)^2}{\sqrt{-x^2+6x-5}}dx$	1 May 1994 9 May 1997 10 August 1997
2	Evaluate the integral	$\int \frac{1}{x\sqrt{x^2-2x}}dx$	2 May 1995
3	Evaluate the integral	$\int \frac{(x-1)}{(x^2-2x-3)^{\frac{3}{2}}}dx$	3 August 1995
4	Evaluate the integral	$\int \frac{x}{\sqrt{x^2+4x+8}}dx$	4 December 1995
5	Evaluate the integral	$\int \frac{x}{\sqrt{x^2+2x}}dx$	5 May 1996 15 December 1998
6	Evaluate the integral	$\int \frac{(x-3)^3}{\sqrt{6x-x^2-8}}dx$	6 July 1996
7	Evaluate the integral	$\int \frac{x}{(x^2+6x+10)^{\frac{3}{2}}}dx$	7 November 1996
8	Evaluate the integral	$\int \frac{x}{\sqrt{2x-x^2}}dx$	16 May 1999
9	Evaluate the integral	$\int \frac{1}{(x+1)\sqrt{12-8x-4x^2}}dx$	17 July 1999
10	Evaluate the integral	$\int \frac{x}{\sqrt{x^2-4x+5}}dx$	19 May 2000
11	Evaluate the integral	$\int \frac{e^x}{e^{2x}-4e^x+8}dx$	20 April 2000
12	Evaluate the integral	$\int \frac{e^x}{\sqrt{9-8e^x-e^{2x}}}dx$	24 August 2001
13	Evaluate the integral	$\int \frac{x}{(x^2+8x+7)^{\frac{3}{2}}}dx$	26 July 2002
14	Evaluate the integral	$\int \sqrt{6x-x^2-8}\,dx$	27 December 2002

Homework

1	Evaluate the integral	$\int \frac{2x+1}{\sqrt{6x-x^2}}dx$	31 December 2003
2	Evaluate the integral	$\int \sqrt{-x^2+4x+5}\ dx$	34 July 2004
3	Evaluate the integral	$\int \frac{2x-1}{x^2-2x+5}dx$	39 December 2005
4	Evaluate the integral	$\int \frac{(x+2)^2}{\sqrt{-x^2-4x}}dx$	45 December 2007
5	Evaluate the integral	$\int \frac{x}{\sqrt{8+2x-x^4}}dx$	46 July 2008
6	Evaluate the integral	$\int \frac{1}{\sqrt{x^2+2x+10}}dx$	
7	Evaluate the integral	$\int \frac{1}{\sqrt{-x^2+10x-21}}dx$	
8	Evaluate the integral	$\int \sqrt{-x^2+4x+5}\,dx$	
9	Evaluate the integral	$\int \frac{1}{(x^2+8x+7)^{\frac{3}{2}}}dx$	
10	Evaluate the integral	$\int \frac{\sqrt{\sin^2 x-2\sin x-8}}{\tan x-\sec x}dx$	
11	Evaluate	$\int \frac{1}{(x-2)\sqrt{x^2-4x+3}}dx$	

Integrals expressions	Form & Substitution	EXAMPLE
Involving quadratic مقدار ثلاثي $ax^2 + bx + c$	(1) rewrit the expressions as (1) ضع المقدار على الصورة $ax^2 + bx + c$ $= a(x+m)^2 + n$ (2) ثم قم بالتعويض التالي (2) Use the substitute $t = x + m \qquad dt = dx$	$I = \int \frac{21}{x^2 - 2x + 5}$ $x^2 - 2x + 5$ $= (x^2 - 2x + 1) - 1 + 5$ $= (x-1)^2 + 4$ $\therefore I = \int \frac{21}{(x-1)^2 + 4}$ Let $t = x - 1 \quad \therefore\ dt = dx$ $\therefore I = \int \frac{21}{t^2+4} = \frac{21}{2}\tan^{-1}\left(\frac{t}{2}\right) + C$ $= \frac{21}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C$

* Evaluate $\int \frac{1}{(x-2)\sqrt{x^2 - 4x + 3}}\,dx$

Solution

$x^2 - 4x + 3 = (x^2 - 4x + 4) - 4 + 3 = (x-2)^2 - 1$

$I = \int \frac{1}{(x-2)\sqrt{x^2-4x+3}}\,dx = \int \frac{1}{(x-2)\sqrt{(x-2)^2 - 1}}\,dx$

Let $t = x - 2 \qquad \Rightarrow dx = dt$

$I = \int \frac{1}{t\sqrt{t^2-1}}\,dx = \sec^{-1} t + c = \sec^{-1}(x-2) + c$